TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 3 2020

Section 2 Calculator Assumed
Systems of Equations and Vector Calculus

STUDENT'S NAME Solutions (PRESSER)

DATE: Friday 22 May **TIME:** 50 minutes **MARKS:** 50

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (6 marks)

Consider the following system of equations. Note: k is a real constant.

$$x - 2y + 3z = 5 \quad \text{— ①}$$
$$x + 2y + z = 5 \quad \text{— ②}$$
$$2x - 4y + kz = 2 \quad \text{— ③}$$

The solutions to the system of equations are:

$x = \dfrac{5k-14}{k-6}$, $y = \dfrac{-4}{k-6}$, $z = \dfrac{A}{k-6}$ where A is a constant.

(a) Explain whether this system of equations has a unique solution for all real values of k. If not, explain the geometric interpretation of this. [3]

Not a unique solution when $k = 6$

If $k = 6$, plane ① and plane ③ are parallel.

(b) Using the third equation, determine the value of A. [3]

Sub values for x, y & z into ③

$$\Rightarrow 2\left(\frac{5k-14}{k-6}\right) - 4\left(\frac{-4}{k-6}\right) + k\left(\frac{A}{k-6}\right) = 2$$

$$\Rightarrow 10k - 28 + 16 + kA = 2k - 12$$

$$A = -8$$

2. (10 marks)

A particle moves such that its position vector is given by $\underset{\sim}{\mathbf{r}}(t) = \begin{pmatrix} t-5 \\ t^2+1 \end{pmatrix}$ for $t \geq 0$.

(a) When does the particle cross the y-axis? [2]

when $x = 0$

i.e. $t - 5 = 0$

$\therefore t = 5$

(b) What is the least distance between the particle and the x-axis? [2]

use $\underset{\sim}{j}$ component

i.e. $y = t^2 + 1$ This has a minimum of 1

$\therefore$ min distance is 1

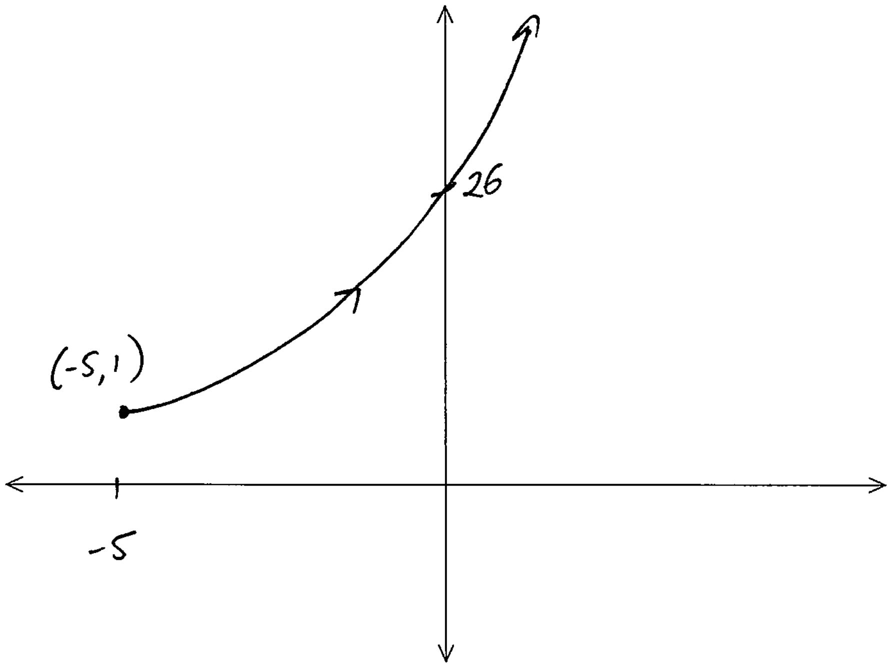
(c) How far is the particle from the origin when $t = 1$? [2]

$\underset{\sim}{r}(1) = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

distance from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is $\sqrt{(-4)^2 + 2^2} = \sqrt{20}$

$= 2\sqrt{5}$

(d) Sketch the path of the particle showing its direction. [3]



3. (8 marks)

The velocity vector $\underset{\sim}{v}(t)\ ms^{-1}$ of a particle is given by $\underset{\sim}{v}(t) = (4\sin 2t)\underset{\sim}{i} - (3\cos 2t)\underset{\sim}{j}$.

(a) Determine the displacement vector $\underset{\sim}{r}(t)$ given that $\underset{\sim}{r}(0) = -2\underset{\sim}{i}$. [2]

$\underset{\sim}{r}(t) = \langle -2\cos 2t \ , -\frac{3}{2}\sin 2t \rangle + \underset{\sim}{c}$

now $\langle -2, 0 \rangle = \underset{\sim}{r}(0) \Rightarrow \langle -2, 0 \rangle = \langle -2, 0 \rangle + \underset{\sim}{c}$

$\therefore \underset{\sim}{r}(t) = \langle -2\cos 2t \ , \frac{3}{2}\sin 2t \rangle$

(b) When will the particle next have a displacement of $-2\mathbf{i}\ m$? [2]

$-2 = -2\cos 2t$

$\Rightarrow \cos 2t = 1$

$2t = 0, 2\pi, \ldots$

$t = 0, \pi, \ldots$

Also check $\underset{\sim}{j}$

$0 = -\frac{3}{2}\sin 2t$

$\sin 2t = 0$

$2t = 0, \pi, 2\pi, \ldots$

$t = 0, \pi/2, \pi \ldots$

$\Rightarrow \therefore t = \underline{\pi}$

(c) Determine the acceleration vector $\underset{\sim}{a}(t)$. [2]

$\underset{\sim}{a}(t) = \langle 8\cos 2t \ , 6\sin 2t \rangle \ m/s^2$

(d) Determine the times, in one complete cycle, when the direction of the particle is moving perpendicular to the acceleration vector. [3]

$\underset{\sim}{a} \cdot \underset{\sim}{v} = \begin{pmatrix} 8\cos 2t \\ 6\sin 2t \end{pmatrix} \cdot \begin{pmatrix} 4\sin 2t \\ -3\cos 2t \end{pmatrix}$

$\Rightarrow 0 = 32\sin 2t\cos 2t - 18\sin 2t$

$\Rightarrow \sin 2t = 0 \qquad \cos 2t = 0$

$t = 0, \pi/2, \pi \qquad t = \pi/4, 3\pi/4$

$\therefore t = 0, \pi/4, \pi/2, 3\pi/4, \pi$ seconds

4. (11 marks)

In the Winter Olympics, a figure skater executed a manoeuvre which could best be described as an ellipse. His ~~acceleration~~ velocity while completing his part of the routine was given by;

$$\underset{\sim}{\mathbf{v}}(t) = \begin{pmatrix} -6\sin\frac{t}{4} \\ 4\cos\frac{t}{4} \end{pmatrix} ms^{-1}$$

His position on the skating rink at the start of his routine is $\underset{\sim}{\mathbf{r}}(0) = \begin{pmatrix} 24 \\ 0 \end{pmatrix}$.

(a) Determine the position of the skater at time t [2]

$$\underset{\sim}{r}(t) = \begin{pmatrix} 24\cos\frac{t}{4} \\ 16\sin t/4 \end{pmatrix} + \underset{\sim}{c} \qquad \text{now } \begin{pmatrix} 20 \\ 0 \end{pmatrix} = \begin{pmatrix} 24 \\ 0 \end{pmatrix} + \underset{\sim}{c}$$

$$\therefore \underset{\sim}{r}(t) = \begin{pmatrix} 24\cos\frac{t}{4} \\ 16\sin t/4 \end{pmatrix}$$

(b) If the skater finishes the ellipse when he returns to the starting position, how long has it taken him to complete the manoeuvre? (leave your answer as an exact value) [2]

returns to $\begin{pmatrix} 24 \\ 0 \end{pmatrix}$

ie. $\begin{pmatrix} 24 \\ 0 \end{pmatrix} = \begin{pmatrix} 24\cos t/4 \\ 16\sin t/4 \end{pmatrix}$

$\Rightarrow \quad \cos t/4 = 1 \qquad \sin t/4 = 0$

$\quad t/4 = 2\pi \qquad t/4 = \cancel{\pi}, 2\pi$

$\therefore \quad t = 8\pi$ seconds

(c) (i) Prove that the skater's speed can be expressed as

$$Speed = a\sqrt{b - k\cos^2\left(\frac{t}{4}\right)}$$, where a, b and k are positive integer constants. [4]

$$Speed = |\underset{\sim}{v}(t)|$$

$$= \sqrt{(-6\sin t/4)^2 + (4\cos t/4)^2}$$

$$= \sqrt{36\sin^2 t/4 + 16\cos^2 t/4}$$

$$= \sqrt{36(1-\cos^2 t/4) + 16\cos^2 t/4}$$

$$= \sqrt{36 - 20\cos^2 t/4}$$

$$= \sqrt{4(9 - 5\cos^2 t/4)}$$

$$= 2\sqrt{9 - 5\cos^2 t/4}$$

(ii) Hence, determine the time(s) at which his speed is the greatest for one complete manoeuvre. State this maximum speed. [3]

Max speed when

$$5\cos^2 t/4 = 0$$

$$\Rightarrow \cos t/4 = 0$$

$$t/4 = \pi/2,\ 3\pi/2$$

$$t = 2\pi,\ 6\pi \text{ seconds}$$

Max speed is $2\sqrt{9} = 6$ m/s

5. (15 marks)

The acceleration of a particle projected with speed u at an angle of elevation of α is given by

$$\underset{\sim}{\mathbf{a}}(t) = \begin{pmatrix} 0 \\ -g \end{pmatrix} \text{ with } \underset{\sim}{\mathbf{r}}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \underset{\sim}{\mathbf{v}}(0) = \begin{pmatrix} u\cos\alpha \\ u\sin\alpha \end{pmatrix}$$

(a) (i) Determine the velocity and displacement vectors $\underset{\sim}{\mathbf{v}}(t)$ and $\underset{\sim}{\mathbf{r}}(t)$. [3]

$$\underset{\sim}{v}(t) = \begin{pmatrix} 0 \\ -gt \end{pmatrix} + \underset{\sim}{c} \qquad \text{now} \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \underset{\sim}{c} = \begin{pmatrix} u\cos\alpha \\ u\sin\alpha \end{pmatrix}$$

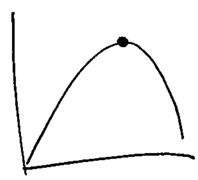
$$\underset{\sim}{v}(t) = \begin{pmatrix} u\cos\alpha \\ u\sin\alpha - gt \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} u\cos\alpha \, t \\ u\sin\alpha \, t - \frac{gt^2}{2} \end{pmatrix} + \underset{\sim}{d} \qquad \text{now} \quad \underset{\sim}{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} u\cos\alpha \, t \\ u\sin\alpha \, t - \frac{gt^2}{2} \end{pmatrix}$$

(ii) Prove that the maximum height attained is $\dfrac{u^2\sin^2\alpha}{2g}$. [3]

Max height when there is no vertical velocity.



$$\Rightarrow \quad u\sin\alpha - gt = 0$$

$$\Rightarrow \quad t = \frac{u\sin\alpha}{g}$$

Max height is j component of $\underset{\sim}{r}(t)$

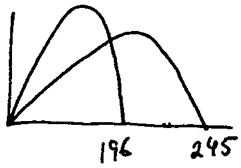
$$\text{max height} = u\sin\alpha\left(\frac{u\sin\alpha}{g}\right) - \frac{g}{2}\left(\frac{u\sin\alpha}{g}\right)^2$$

$$= \frac{2u^2\sin^2\alpha}{2g} - \frac{u^2\sin^2\alpha}{2g}$$

$$= \frac{u^2\sin^2\alpha}{2g}$$

(b) A golf ball is struck so that it leaves point A on the ground with speed 49 ms^{-1} at an angle of elevation α. If it lands on the green which is the same level as A, the nearest and furthest points of which are 196 metres and 245 metres respectively from A. ($g = 9.8$ ms^{-2}).

Given that the horizontal range is given by $\dfrac{u^2 \sin 2\alpha}{g}$



(i) Calculate the set of possible values of α. [3]

$$\frac{49^2 \sin 2\alpha}{9.8} = 196$$

$$2\alpha = 53.13^\circ, 126.87^\circ$$

$$\alpha = 26.6^\circ, 63.4^\circ$$

$$\frac{49^2 \sin 2\alpha}{9.8} = 245$$

$$2\alpha = 90$$

$$\alpha = 45^\circ$$

Max range

$\therefore$ $26.6^\circ \le \alpha \le 63.4^\circ$ to land on green.

(ii) Determine the maximum height the ball can reach and still land on the green. [2]

when $\alpha = 63.4^\circ$

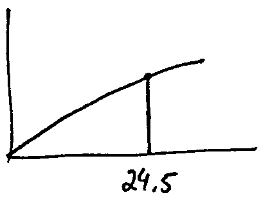
$$\text{max height} = \frac{49^2 \sin^2(63.4^\circ)}{2 \times 9.8}$$

$$= 97.9 \text{ m}$$

There is a tree at a horizontal distance 24.5 metres from A and to reach the green the ball must pass over this tree. (The point A, the green and the base of the tree are in the same horizontal plane)

(iii) Determine the maximum height of the tree if this ball can reach any point on the green. [4]

Lowest angle is 26.6°



$$x = ut\cos\alpha$$

$$\Rightarrow t = \frac{24.5}{49\cos 26.6^\circ}$$

$$= 0.559 \text{ sec}$$

height $y = 49\sin 26.6^\circ(0.559) - 4.9(0.559)^2$

$$= 10.73 \text{ m}$$